

**FIXED POINT AND RELATED GEOMETRIC PROPERTIES
ON THE FOURIER AND FOURIER STIELTJES ALGEBRAS
OF LOCALLY COMPACT GROUPS**

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Outline of Talk

- Historical remarks
- Weak fixed point property and Radon Nikodym property on preduals of von Neumann algebras
- Weak fixed point property of the Fourier algebra
- Fixed point property of the Fourier algebra
- Weak* fixed point property for the Fourier Stieltjes algebra:

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Let K be a bounded closed convex subset of a Banach space. A mapping $T : K \rightarrow K$ is called **non-expansive** if

$$\|T(x) - T(y)\| \leq \|x - y\|, \quad x, y \in K.$$

In general, K need **NOT** contain a fixed point for T :

Example 1. $E = c_0$: all sequences (x_n) , $x_n \in \mathbb{R}$, such that $x_n \rightarrow 0$

$$\|(x_n)\| = \sup \{|x_n|\}.$$

Define: $T(x_1, x_2, \dots) = (1, x_1, x_2, \dots)$

$$K = \text{unit ball of } c_0.$$

Then T is a non-expansive mapping $K \rightarrow K$ without a fixed point.

Example 2. $E = \ell^1$: all sequences (x_n) such that

$$\sum |x_n| < \infty$$

$$\|x_n\|_1 = \sum |x_n|.$$

Let $S : \ell^1 \rightarrow \ell^1$ be the shift operator:

$$S(x_n) = (0, x_1, x_2, \dots)$$

$$K = \{(x_n) : x_n \geq 0, \|x_n\|_1 = 1\}.$$

Then S is a non-expansive mapping $K \rightarrow K$ without a fixed point.

Proposition. *Let K be a bounded closed convex subset of a Banach space, and $T : K \rightarrow K$ is non-expansive, then T has an approximate fixed point, i.e. \exists a sequence $x_n \in K$ such that $\|T(x_n) - x_n\| \rightarrow 0$.*

Proof: We assume $0 \in K$. For each $1 > \lambda > 0$, define

$$T_\lambda(x) = T(\lambda x).$$

Then

$$\begin{aligned} \|T_\lambda(x) - T_\lambda(y)\| &= \|T(\lambda x) - T(\lambda y)\| \\ &\leq \|\lambda x - \lambda y\| = \lambda \|x - y\| \end{aligned}$$

so by the Banach Contractive Mapping Theorem, $\exists x_\lambda \in K$ such that $T_\lambda(x_\lambda) = x_\lambda$.

Now

$$\begin{aligned}\|T(x_\lambda) - x_\lambda\| &= \|T(x_\lambda) - T_\lambda(x_\lambda)\| \\ &= \|T(x_\lambda) - T(\lambda x_\lambda)\| \\ &\leq \|x_\lambda - \lambda x_\lambda\| \\ &= (1 - \lambda)\|x_\lambda\| \rightarrow 0.\end{aligned}$$

Example 3 (Alspach, PAMS 1980)

$$E = L^1[0, 1] \quad \|f\|_1 = \int_0^1 |f(t)| dt$$
$$K = \left\{ f \in L^1[0, 1], \int_0^1 f(x) dx = 1, \quad 0 \leq f \leq 2 \right\}.$$

Then K is **weakly** compact and convex.

$$T : K \rightarrow K$$

$$(Tf)(t) = \begin{cases} \min \{2f(2t), 2\}, & 0 \leq t \leq \frac{1}{2} \\ \max \{2f(2t - 1) - 2, 0\}, & \frac{1}{2} < t \leq 1. \end{cases}$$

Then T is non-expansive, and fixed point free.

Theorem (T. Dominguez-Benavides, M.A. Japon, and S. Prus, J. of Functional Analysis, 2004). *Let C be a nonempty closed convex subset of a Banach space. Then C is weakly compact if and only if C has the **generic** fixed point property for continuous affine maps i.e. if $K \subseteq C$ is a nonempty closed convex subset of C , and $T : K \rightarrow K$*

T is continuous and affine, then T has a fixed point in K .

A map $T : K \rightarrow K$ is **affine** if for any $x, y \in K$, $0 \leq \lambda \leq 1$, $T(\lambda x + (1 - \lambda)y) = \lambda T(x) + (1 - \lambda)T(y)$.

Let X be a bounded closed convex subset of a Banach space E . A point x in X is called a **diametral point** if

$$\sup \{ \|x - y\| : y \in X \} = \text{diam}(X).$$

The set X is said to have **normal structure** if every nontrivial (i.e. contains at least two points) convex subset K of X contains a non-diametral point.

Theorem (Kirk, 65). *If X is a weakly compact convex subset of E , and X has normal structure, then every non-expansive mapping $T : X \mapsto X$ has a fixed point.*

Remark:

1. compact convex sets always have normal structure.
2. Alspach's example shows that weakly compact convex sets need **not** have normal structure.

A Banach space E is said to have the **weak fixed point property** (weak-f.p.p.) if for each weakly compact convex subset $X \subseteq E$, and $T : X \rightarrow X$ a non-expansive mapping, X contains a fixed point for T .

Theorem (F. Browder, 65). *If E is uniformly convex, then E has the weak fixed point property.*

Theorem (B. Maurey, 81). *c_0 has the weak fixed point property.*

Theorem (T.C. Lim, 81). *ℓ_1 has the weak* fixed point property and hence the weak fixed point property.*

Theorem (Llorens - Fusta and Sims, 1998).

- *Let C be a closed bounded convex subset of c_0 . If the set C has an interior point, then C fails the weak f.p.p.*
- *There exists non-empty convex bounded subset which is compact in a locally convex topology slightly coarser than the weak topology and fails the weak f.p.p.*

Question: Does weak f.p.p. for a closed bounded convex set in c_0 characterize the set being weakly compact?

Theorem (Dowling, Lennard, Turrett, Proceedings A.M.S. 2004). *A non-empty closed bounded convex subset of c_0 has the weak f.p.p. for non-expansive mapping \iff it is weakly compact.*

Radon Nikodym Property and Weak Fixed Point Property

Banach space E is said to have **Radon Nikodym property** (RNP) if each closed bounded convex subset D of E is dentable i.e. for any $\varepsilon > 0$, there exists and $x \in D$ such that $x \notin \overline{\text{co}}(D \setminus B_\varepsilon(x))$, where

$$B_\varepsilon(x) = \{y \in E; \|y - x\| < \varepsilon\}.$$

Theorem (M. Rieffel). *Every weakly compact convex subset of a Banach space is dentable.*

Note: 1. $L^1[0, 1]$ does not have f.p.p and R.N.P.

2. ℓ^1 has the f.p.p. and R.N.P.

Question: Is there a relation between f.p.p. and R.N.P.?

Theorem 1 (Mah-Ülger-Lau, PAMS 1997). *Let M be a von Neumann algebra. If M_* has the RNP, then M_* has the weak f.p.p.*

Note: c_0 has the weak f.p.p. but not the R.N.P.

However $c_0 \not\cong M_*$, M a von Neumann algebra.

$M =$ von Neumann algebra

$$\subseteq B(H)$$

$M^+ =$ all positive operators in M

$\tau : M^+ \rightarrow [0, \infty]$ be a trace i.e. a function on M^+ satisfying:

- (i) $\tau(\lambda A) = \lambda \tau(A)$, $\lambda \geq 0$, $A \in M^+$
- (ii) $\tau(A + B) = \tau(A) + \tau(B)$, $A, B \in M^+$
- (iii) $\tau(A^*A) = \tau(AA^*)$ for all $A \in M$

τ is **faithful** if $\tau(A) = 0$, $A \in M^+$, then $A = 0$.

τ is **semifinite** if $\tau(A) = \sup\{\tau(B); B \in M^+, B \leq A, \tau(B) < \infty\}$.

τ is **normal** if for any increasing net $(A_\alpha) \subseteq M^+$,

$A_\alpha \uparrow A$ in the weak*-topology, then $\tau(A_\alpha) \uparrow \tau(A)$.

Theorem 2 (Leinert - Lau, TAMS 2008). *Let M be a von Neumann algebra with a faithful normal semi-finite trace, then M_* has RNP $\iff M_*$ has the weak f.p.p.*

$G =$ locally compact group with a fixed left Haar measure λ .

- A continuous unitary representation of G is a pair: $\{\pi, H\}$, where $H =$ Hilbert space and π is a continuous homomorphism from G into the group of unitary operators on H such that for each $\xi, \eta \in H$,

$$x \rightarrow \langle \pi(x)\xi, \eta \rangle$$

is continuous.

- $\{\pi, H\}$ is *irreducible* if $\{0\}$ and H are the only $\pi(G)$ -invariant subspaces of H .
- $\{\pi, H\}$ is *atomic* if $\{\pi, H\} \cong \sum \oplus \{\pi_\alpha, H_\alpha\}$ where each π_α is a irreducible representation.

$$L^2(G) = \text{all measurable } f : G \rightarrow \mathbb{C}$$

$$\int |f(x)|^2 d\lambda(x) < \infty$$

$$\langle f, g \rangle = \int f(x) \overline{g(x)} d\lambda(x)$$

$L^2(G)$ is a Hilbert space.

Left regular representation:

$$\{\rho, L^2(G)\},$$

$$\rho : G \mapsto B(L^2(G)),$$

$$\rho(x)h(y) = h(x^{-1}y), \quad x \in G, \quad h \in L^2(G).$$

$G =$ locally compact group

$A(G) =$ Fourier algebra of G

$=$ subalgebra of $C_0(G)$

consisting of all functions $\phi :$

$$\phi(x) = \langle \rho(x)h, k \rangle, \quad h, k \in L^2(G)$$

$$\rho(x)h(y) = h(x^{-1}y)$$

$$\begin{aligned} \|\phi\| &= \sup \left\{ \left| \sum_{i=1}^n \lambda_i \phi(x_i) \right| : \left\| \sum_{i=1}^n \lambda_i \rho(x_i) \right\| \leq 1 \right\} \\ &\geq \|\phi\|_\infty. \end{aligned}$$

P. Eymard (1964):

$$\begin{aligned} A(G)^* &= VN(G) \\ &= \text{von Neumann algebra in } \mathcal{B}(L^2(G)) \\ &\quad \text{generated by } \{\rho(x) : x \in G\} \\ &= \overline{\langle \rho(x) : x \in G \rangle}^{\text{WOT}} \end{aligned}$$

If G is abelian and \widehat{G} = dual group of G , then

$$A(G) \cong L^1(\widehat{G}), \quad VN(G) \cong L^\infty(\widehat{G})$$

When G is **abelian**, $\widehat{G} =$ dual group

$$\mathbb{T} = \{\lambda \in \mathbb{C}, |\lambda| = 1\}$$

$$\widehat{\mathbb{T}} = (\mathbb{Z}, +), \quad \widehat{\mathbb{Z}} = \mathbb{T}.$$

Hence $A(\mathbb{Z}) \cong L^1(\mathbb{T})$.

Theorem (Alspach). *If $G = (\mathbb{Z}, +)$, then $A(\mathbb{Z})$ does **not** have weak f.p.p.*

Question: Given a locally compact group G , when does $A(G)$ have the weak f.p.p.?

Theorem (Mah - Lau, TAMS 1988). *If G is a compact group, then $A(G)$ has the weak f.p.p.*

Theorem (Mah - Ülger - Lau, PAMS 1997).

a) *If G is abelian, then $A(G)$ has the weak f.p.p. $\iff G$ is compact.*

b) *If G is discrete and $A(G)$ has the weak f.p.p., then G cannot contain an infinite abelian subgroup. In particular, each element in G must have finite order.*

Example: $G =$ all 2×2 matrices

$$\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \longleftrightarrow (x, y)$$

with $x, y \in \mathbb{R}$, $x \neq 0$. (“ $ax + b$ ”-group).

Topologize G as a subset of \mathbb{R}^2 with multiplication

$$(x, y) \circ (u, v) = (xu, xv + y).$$

Then G is a non-compact group. But $A(G)$ has Radon Nikodym Property (K. Taylor). Hence it must have weak f.p.p.

A locally compact group G is called an [IN]-group if there is a compact neighborhood U of the identity e such that $x^{-1}Ux = U$ for all $x \in G$.

Example: compact groups
discrete groups
abelian groups

Theorem 3 (Leinert - Lau, TAMS 2008). *Let G be an [IN]-group. TFAE:*

- (a) G is compact
- (b) $A(G)$ has weak f.p.p.
- (c) $A(G)$ has RNP

Corollary. *Let G be a discrete group. Then $A(G)$ has the weak f.p.p. $\iff G$ is finite.*

Proof: If G is a [SIN]-group, then $VN(G)$ is finite. Apply Theorems 1 and 2.

- A (discrete) semigroup S is left reversible if $aS \cap bS \neq \emptyset$ for any $a, b \in S$.
- S commutative $\implies S$ is left reversible.
- We say that a Banach space E has the weak f.p.p. for commutative (left reversible) semigroup if whenever S is a commutative (resp. left reversible) semigroup and K is a weakly compact convex subset of E for on K and $\mathcal{S} = \{T_s : s \in S\}$ is a representation of S as non-expansive mappings from K into K , then K has a common fixed point for \mathcal{S} .

Theorem (R. Bruck, 74). *If a Banach space E has the weak f.p.p., then E has the weak f.p.p. for commutative semigroup.*

Corollary. *If G is a locally compact group such that $A(G)$ has the RNP, then $A(G)$ has the weak fixed point property for commutative semigroups.*

Theorem 4 (Lau-Mah, JFA 2010). *Let G be an $[\text{IN}]$ -group TFAE.*

- (a) *G is compact.*
- (b) *$A(G)$ has the weak f.p.p. for left reversible semigroup.*

Theorem (Garcia-Falset). *If H is a Hilbert space $\mathcal{K}(H) = C^*$ -algebra of compact operators on H has the weak fixed point property.*

- If G is a compact group, then

$$C^*(G) = \overline{\{\rho(f); f \in L^1(G)\}} \subseteq \mathcal{K}(L^2(G)).$$

Hence $C^*(G)$ has the weak fixed point property. Consequently the weak fixed point property for commutative semigroups.

Problem 1. If G is a compact group, does $C^*(G)$ have the weak f.p.p. for left reversible semigroups?

Proposition (Lau-Mah-Ülger, PAMS 1997). *$VN(G)$ has the weak f.p.p. for left reversible semigroup if and only if G is finite.*

Problem 2 (Bruck). If a Banach space E has the weak f.p.p., does it always have the weak f.p.p. for left reversible (or amenable) semigroup?

Fixed Point Property

Let E be a Banach space, and K be a non-empty bounded closed convex subset of E . We say that K has the **fixed point property (f.p.p.)** if every nonexpansive mapping $T : K \rightarrow K$ has a fixed point. We say that E has the fixed point property if every bounded closed convex subset K of E has the fixed point property.

- ℓ^p , $1 < p < \infty$, has the fixed point property
- ℓ^1 has the weak fixed point property but not the fixed point property
- A closed subspace of $L^1[0, 1]$ has the fixed point property if and only if it is reflexive.

Theorem 5 (Leinert and Lau, TAMS 2008). *For G locally compact, if a nonzero closed ideal of A has the f.p.p., then G is discrete.*

Corollary. *$A(G)$ has the f.p.p. $\iff G$ is finite.*

Proof. By above, G must be discrete. Since f.p.p. \implies weak f.p.p., it follows that $A(G)$ has the weak f.p.p. Consequently, it must be finite.

Theorem (P.K. Lin, Nonlinear Analysis 2008). ℓ^1 can be renormed to have the f.p.p.

Theorem (C. Hernandez Lineares and M.A. Japon, JFA 2010). If G is a separable compact group, then $A(G)$ can be renormed to have the f.p.p.

Remark (Dowling, Lennard and Turett, TMAA 1996): This theorem is not true for non-separable groups.

Weak* Fixed Point Property

A dual Banach space E is said to have **weak*-f.p.p.** if every weak*-compact convex subset K of E has the fixed point property.

E is said to have the **weak* Kadec-Klee property** if the weak*-topology and norm topology agree on the unit sphere.

Theorem (T.C. Lim, Pacific J. Math. 1980). $\ell_1 = c_0^*$ has the weak*-f.p.p. property.

Theorem (C. Lennard, PAMS 1990). Let H be a Hilbert space. Then $B(H)_*$ has the weak*-f.p.p.

G -locally compact group

$P(G) =$ continuous positive definite

functions on G

i.e. all continuous $\phi : G \rightarrow \mathbb{C}$ such that

$$\sum \lambda_i \bar{\lambda}_j \phi(x_i x_j^{-1}) \geq 0, \quad \begin{array}{l} x_1, \dots, x_n \in G, \\ \lambda_1, \dots, \lambda_n \in \mathbb{C} \end{array}$$

i.e. the $n \times n$ matrix $(\phi(x_i x_j^{-1}))$ is positive

$\phi \in P(G) \iff$ there exists a continuous

unitary representation $\{\pi, \mathcal{H}\}$

of G , $\eta \in \mathcal{H}$, such that

$$\phi(x) = \langle \pi(x)\eta, \eta \rangle, \quad x \in G.$$

Let $B(G) = \langle P(G) \rangle \subseteq CB(G)$ (**Fourier Stieltjes algebra** of G)

Equip $B(G)$ with norm $\|u\| = \sup \{ |\int f(t)u(t)dt|; f \in L^1(G) \text{ and } |||f||| \leq 1 \}$

where

$$|||f||| = \sup\{ \|\pi(f)\|; \{\pi, H\} \text{ continuous unitary representation of } G \}$$

Let $C^*(G)$ denote the completion of $(L^1(G), |||\cdot|||)$. Then $C^*(G)$ is a C^* -algebra (the **group C^* -algebra** of G), and $B(G) = C^*(G)^*$.

- When G is *amenable*, then $|||f||| = \|\rho(f)\|$, where ρ is the left regular representation of G .
- When G is abelian, $B(G) \cong M(\widehat{G})$ (measure algebra of \widehat{G}), and $C^*(G) \cong C_0(\widehat{G})$.

A dual Banach space E is said to have the *weak* -Kadec-Klee property* if the norm and weak* -topology agree on $E_1 = \{x \in E; \|x\| = 1\}$.

Theorem (Lau-Mah, TAMS 88). (a) *For a locally compact group G , the measure algebra $M(G)$ has the weak* fpp $\iff G$ is discrete $\iff M(G)$ has the weak*-Kadec-Klee property.*

(b) *If G is compact, then $B(G) = C^*(G)^*$ has the weak*-fpp.*

Theorem (Lau-Mah, TAMS 88/Bekka-Kaniuth-Lau-Schlichting, TAMS 1998). *Let G be a locally compact group. Then G is compact $\iff B(G)$ has the weak* Kadec Klee property.*

Theorem 6 (Fendler-Lau-Leinert, JFA 2013). *If G is a locally compact group and $B(G)$ has the w^* -f.p.p. then G is compact.*

Theorem (T.C. Lim, Pacific J. Math. 1980). *The dual Banach space $B(\mathbb{T}) \cong \ell^1(\mathbb{Z})$ has the weak* f.p.p. for left reversible semigroup.*

Theorem 7 (Fendler-Lau-Leinert, JFA 2013). *For any compact group G , $B(G)$ has the weak* f.p.p. for left reversible semigroups.*

When G is separable, Theorem 6 and Theorem 7 were proved by Lau and Mah (JFA, 2010).

Key Lemma

Lemma A. *Let G be a compact group, and let $\{D_\alpha : \alpha \in \Lambda\}$ be a decreasing net of bounded subsets of $B(G)$, and $\{\phi_m : m \in M\}$, be a weak* convergent bounded net with weak* limit ϕ . Then*

$$\limsup_m \limsup_\alpha \{\|\phi_m - \psi\| : \psi \in D_\alpha\} = \limsup_\alpha \{\|\phi - \psi\| : \psi \in D_\alpha\} \\ + \limsup_m \|\phi_m - \phi\|.$$

Let C be a nonempty subset of a Banach space X and $\{D_\alpha : \alpha \in \Lambda\}$ be a decreasing net of bounded nonempty subsets of X . For each $x \in C$, and $\alpha \in \Lambda$, let

$$r_\alpha(x) = \sup \{\|x - y\| : y \in D_\alpha\},$$

$$r(x) = \lim_{\alpha} r_\alpha(x) = \inf_{\alpha} r_\alpha(x),$$

$$r = \inf \{r(x) : x \in C\}.$$

The set (possibly empty)

$$\mathcal{AC}(\{D_\alpha : \alpha \in \Lambda\}) = \{x \in C : r(x) = r\}$$

is called the *asymptotic center* of $\{D_\alpha : \alpha \in \Lambda\}$ with respect to C and r is called the *asymptotic radius* of $\{D_\alpha : \alpha \in \Lambda\}$ with respect to C .

(M. Edelstein, Bull. A.M.S. 1972)

Theorem 8 (Fendler-Lau-Leinert, JFA 2013). *Let G be a compact group. Let C be a nonempty $weak^*$ closed convex subset of $B(G)$ and $\{D_\alpha : \alpha \in \Lambda\}$ be a decreasing net of nonempty bounded subsets of C . Let $r(x)$ be as defined above. Then for each $s \geq 0$, $\{x \in C : r(x) \leq s\}$ is $weak^*$ compact and convex, and the asymptotic center of $\{D_\alpha : \alpha \in \Lambda\}$ with respect to C is a nonempty norm compact convex subset of C .*

Theorem (Narcisse Randrianantoania, JFA 2010). *For any G :*

(a) *$A(G)$ has the weak f.p.p. $\iff A(G)$ has the R.N.P. \iff The left regular representation of G is atomic. In this case $A(G)$ has the weak f.p.p. for left reversible semigroups.*

(b) *$B(G)$ has the weak f.p.p. $\iff B(G)$ has R.N.P. \iff every continuous unitary representation of G is atomic. In this case $B(G)$ has the weak f.p.p. for left reversible semigroups.*

Theorem 8 answers the following problem: For any locally compact group G does R.N.P. on $B(G)$ imply weak* f.p.p.?

Open problem 3. Let G be a locally compact group. Let $B_\rho(G)$ denote the reduced Fourier-Stieltjes algebra of $B(G)$, i.e. $B_\rho(G)$ is the weak* closure of $C_{00}(G) \cap B(G)$. Then $B_\rho(G) = C_\rho(G)^*$. Does the weak* fixed point property on $B_\rho(G)$ imply G is compact? This is true when G is amenable by Theorem 6, since $B(G) = B_\rho(G)$ in this case.

Open problem 4. Let G be a locally compact group. Does the asymptotic centre property on $B_\rho(G)$ imply that G is compact?

Theorem (Fendler and Leinert, 2013). *For any von Neumann algebra, if M_* has the RNP, then M_* has the weak fixed point property.*

Problem: When G is a topological group,

$P(G)$ = continuous positive definite functions on G

$B(G)$ = linear span of $P(G)$.

Theorem (Lau-Ludwig, Advances of Math 2012). $B(G)^*$ is a von Neumann algebra.

Problem 5: When does $B(G)$ have the weak fixed point property?

APPENDIX A

A Banach space X is said to be **uniformly convex** if for each $0 < \varepsilon \leq 2$, $\exists \delta > 0$ such that for any $x, y \in X$,

$$\left. \begin{array}{l} \|x\| \leq 1 \\ \|y\| \leq 1 \\ \|x - y\| > \varepsilon \end{array} \right\} \left\| \frac{x + y}{2} \right\| \leq \delta$$

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